Mathematics on the Board

Basics

Vinay Iyer

ISBN: 978-93-47365-01-0

Title: Mathematics on the board: Basics

Author: Vinay Iyer

Date of publication: 27th August, 2026

Publisher:
Divamani Enterprises,
Ernakulam, India
Phone no.: +91 62382 89930
Email: divamani.enterprises@divamani.in

Introduction

"Mathematics on the board" is a literal translation of the Sanskrit phrase *pāṭīgaṇita*. For most of Indian history, calculations that were meant to be temporary, were written on a wooden board, before the results were copied onto more permanent writing media such as palm leaves or bark sheets. This is the origin of the name *pāṭīgaṇita* (mathematics done on the board) — by extension, it denotes the basic lessons of arithmetic, geometry and related portions of mathematics that was taught to a student during their earliest years.

Indian mathematics has a long history — much of it dates back at least 3,000 years if not more. We often hear about how Indians "discovered" zero or about the decimal place value system being an Indian innovation. However, it begs the question: is that all that Indian mathematics has to offer? What makes Indian mathematics unique and relevant, at a time when Western mathematics has advanced far?

Here are some of the unique features of Indian mathematics and why they become relevant to any student even today:

- **Algorithms instead of equations**: Indian mathematics does not use Western-style equations. It instead places all heavy work on algorithms. Many students struggle with mathematics and sciences when they encounter a scary-looking equation like: $F = \frac{GMm}{R^2}$ but have no problem following a step-by-step procedure such as a recipe to cook a dish. They will find in Indian mathematics, simple methods that suit their needs and tastes.
- **Comfort with large numbers**: Indians did not use decimals, and used fractions in their place. This meant that they often had to deal with large numbers, and this pushed innovation for computing and working with these with ease, even though everything was done by hand.
- **Simple methods for complex operations**: Since computations were largely done by hand, there was always a push for reducing complex operations and simplifying whatever was there, as much as possible.
- **Practical, not abstract**: Much of Western mathematics since Greek times has been influenced by the use of axioms, theorems and statements of truth. Indians, on the other hand, were more concerned with quick computations that achieved a result within practical requirements of

accuracy. While similar goals did and still do exist in Western sciences and engineering, potential for any conflict was largely done away within the Indian system from the beginning. This gives us a cohesive methodology that is focussed and geared towards solving problems quickly. Mathematics is ultimately a tool, and using it effectively is a necessity.

There are several more unique features that will become apparent when a student studies the subject and brings the principles of this ancient tradition into practice.

This book presents the most elementary portions of *pāṭīgaṇita* in simple language. It only assumes knowledge of school-level mathematics — everything else is introduced from basic principles. It does not delve into the history of Indian mathematics, although that is indeed worth studying by itself. Sanskrit was the language through which educated audiences throughout India communicated with each other, hence technical works in most subjects tend to either be written in Sanskrit, or peppered with Sanskrit words. While "Indian" does indeed cover all the languages of India, knowledge of Sanskrit would be essential for a student who wanted to access the pan-Indian system easily. For this reason, this book uses Sanskrit terminology for technical terms.

Examples have been provided in all but the most basic concepts. Each chapter ends with a set of questions that would help any reader to practice the methods given here. However, a reader who is interested in developing this further should try to apply it in their daily life wherever possible, since practice is always beneficial in mathematics.

A glossary of Sanskrit technical terms is included at the end. Many common Sanskrit words such as *phala* (fruit) have specific technical meaning in mathematics (*phala* means 'result'). This list supplies these meanings.

Those who have completed this book, and who are well versed in Sanskrit may find interest in reading *Līlāvatī*, written by Bhāskara (12th century CE). Much of the pedagogical approach and the contents of this book are influenced by this book, although the examples and approach have been made suitable for modern audiences where deemed necessary.

It is my sincere hope that this book will be of help to any student who truly wishes to know the Indian tradition of mathematics.

Transcription

This book uses the International Alphabet of Sanskrit Transliteration (IAST) scheme to represent Sanskrit text in Roman characters. The following table provides the reference for conversion between Devanagari and IAST.

Dev.	IAST
अ	a
आ	ā
इ	i
ई	ī
उ	u
ऊ	ū
ऋ	ṛ
ॠ	ṝ
ऌ	ḷ
ए	e
ऐ	ai
ओ	o
औ	au
अं	ṃ
अः	ḥ
क	k
ख	kh
ग	g
घ	gh
ङ	ṅ
च	c
छ	ch
ज	j
झ	jh

Dev.	IAST
ञ	ñ
ट	ṭ
ठ	ṭh
ड	ḍ
ढ	ḍh
ण	ṇ
त	t
थ	th
द	d
ध	dh
न	n
प	p
फ	ph
ब	b
भ	bh
म	m
य	y
र	r
ल	l
व	v
श	ś
ष	ṣ
स	s
ह	h

Dev.	IAST
ँ	m̐
ऽ	‘

Contents

Chapter 1

Place Value

At the very heart of mathematics lie the numbers. They represent its breath, and form the basis of everything in it. However, the way in which numbers are represented has varied significantly across cultures. For example, in day-to-day life, we often use Roman numerals — the system originally used in the Roman Empire.

The system followed today, involving 10 different digits (including zero), however, first arose in India.

Much like the modern system, the original Indian place value system used 10 digits — the digits 1-9, along with one digit for 0. The position of the number indicated its value. So, in the number "12", 1 does not mean one but ten. Shifting one place to right meant multiplying by ten, and when no digit is marked, we add a '0' to indicate absence — this was the key insight of the Indian system that made it different from other systems of its time.

Two things are different about the Indian system:

Firstly, each position in the number has a name. In today's terms that would be 'unit's place', 'ten's place' and so on. In Indian treatises, the names of the digits in order from right to left are:

Position	Name	Number
1	Eka	1
2	Daśa	10
3	Śata	100
4	Sahasra	1000
5	Ayuta	10000
6	Lakṣa	100000
7	Prayuta	1000000
8	Koṭi	10000000
9	Arbuda	100000000
10	Abda	1000000000
11	Kharva	10000000000

Position	Name	Number
12	Nikharva	100000000000
13	Mahāpadma	1000000000000
14	Śaṅkha	10000000000000
15	Jaladhi	100000000000000
16	Antya	1000000000000000
17	Madhya	10000000000000000
18	Parārdha	100000000000000000

These names were specifically created to refer to these positions. Though names beyond *parārdha* existed, they were seldom used. Real life use cases usually extended only upto *koṭi*. Hence, it's not surprising that this is the largest number in this list whose name is still remembered by most people in India (crores). The reasoning for counting further upto to *parārdha* is to count specific time periods in the Purāṇas that covered very large timescales.

The second distinctive feature of the Indian system is that there is no decimal point or concept of decimals — we always work with fractions. We will look at fractions in detail in Chapter 3.

Chapter 2

Eight Operations

Indian mathematics recognises eight basic operations as the starting point of mathematics. These are called *parikarmas*. The eight operations are:

1. Addition
2. Subtraction
3. Multiplication
4. Division
5. Square
6. Square root
7. Cube
8. Cube root

Addition

Addition of two numbers is similar to what we have studied in school. The numbers are placed one below the other and the digits of the same place are added individually. Higher digits will also be carried as usual.

The terms related to addition are:
- *yuti*: addition (as in the act of adding two numbers)
- *saṅkalita*: addition (as an operation – this is more abstract)
- *yoga*: sum

Subtraction

Subtraction of two numbers, much like addition, is similar to what we have studied in school. Digits are placed under those of the same place value and subtraction is performed.

The terms connected to subtraction are:
- *viyuti*: subtraction (the act itself)
- *vyavakalita*: subtraction (the operation – this is more abstract)
- *viyuta*: subtracted
- *ūna*: subtracted
- *antara*: difference

Multiplication

Multiplication is called *guṇa* or *guṇakāra*. One general method is similar to what we study in school. However, Indians innovated and worked with several alternatives. Some of the terms connected to multiplication:
- *guṇya* - the number to be multiplied (multiplicand)
- *guṇaka* - the number with which it is multiplied (multiplier)

- *hata* - multiplied
- *āhati* - product
- *saṃvarga* - product
- *abhyāsa* - product

One simple method, especially when working with larger numbers, is the door-joint (*kapāṭasandhi*) method. The steps to multiply two numbers with this are given below:

1. Place the first number left to right in the usual way.
2. Place the second number right to left such that the unit's place of this number sits under the unit's place of the first number.
3. Multiply the two unit's place digits and place the result separately. Ten's digit in the result should be taken as carry.
4. Slide the second number to the left by one place.
5. Multiply the digits that are one below the other and add up what you got.
6. Add the result to the ten's place of what was kept aside. Ten's and hundred's digits of the result are carried.
7. Repeat the sliding until all digits of the second number have slid past the first number.
8. The result that arises on the side (after adding up all carried over digits) is the final answer.

Notice how this appears like the joint of a sliding door? The lower number, much like the door, slides past the upper number, which is like the beam at the top.

Let us look at this through an example: 132×596.

1.

```
1 3 2
    6 9 5
```

$6 \times 2 = 12$
1 is carried.

1
2

2.

```
1 3 2
  6 9 5
```

$3 \times 6 = 18$
$2 \times 9 = 18$
$18 + 18 = 36$

3 1

6 2

3.

1 3 2

6 9 5

$$1 \times 6 = 6$$
$$3 \times 9 = 27$$
$$2 \times 5 = 10$$
$$6 + 27 + 10 = 43$$

4 3 1

3 6 2

4.

1 3 2

6 9 5

$$1 \times 9 = 9$$
$$3 \times 5 = 15$$
$$9 + 15 = 24$$

2 4 3 1

4 3 6 2

5.

1 3 2

6 9 5

$$1 \times 5 = 5$$

2 4 3 1

5 4 3 6 2

Add all the carried digits:

2	4	3	1	
5	4	3	6	2
7	8	6	7	2

The answer is **78672**.

Division

Division of one number by another is the same as what we have studied in school. There are, however, two outcomes:

1. Retain the result as a fraction. For example: $5 \div 3 = \frac{5}{3}$.
2. Divide fully and find quotient and remainder. For example: $5 \div 3$. Here: quotient is 1 and remainder is 2.

Either may be used depending on the context and what is needed.

The terms typically used include:
- *bhakta* - divided
- *hṛta* - divided
- *haraṇa* - division

- *phala* - result in general (can be used with any operation, but is specifically used in this context to denote the quotient)
- *śeṣa* - remainder
- *taṣṭa* - divide and take the remainder

Square

Varga refers to squaring. Squaring is a simple operation — multiply a number by itself. In modern mathematics, squaring is considered as a part of exponents in general, and is therefore not spoken of as a separate operation. However, since it is one of the most heavily used operations in practical contexts, such as in geometry to find area, Indian mathematicians considered it separately.

For small numbers, multiplying the two numbers directly is the easiest way of finding the number's square. However, with large numbers, this can be cumbersome. As this is a very common operation, there is a need for simplifying operations when dealing with very large numbers.

One of the methods commonly used is to split the number into chunks and find squares individually.

Another trick is as follows:
- Split the number into two smaller numbers such that the two are close to each other.
- Find the difference of the chunks.
- Find the square of the difference and keep it in one place.
- Take the two chunks and find their product.
- Multiply the product by 4.
- Add the square of the difference that was kept aside, to this.

The result is the square of the original number.

Let's say we want to compute the square of 102. The steps are as follows:

- Split this into 50 and 52.
- The difference ($52 - 50$) is 2. Its square is 4. Keep it in one place.
- Multiply these numbers together (50×52). This gives 2600.
- Multiply 2600 by 4. We get 10400.
- Add to this the square of the difference which we kept aside ($10400 + 4$).

The final answer is 10404, which is the square of 102. These methods enable quick calculation of the *varga*.

Square Root

Vargamūla refers to square roots. It is composed of two words - *varga*, meaning ‘square’ and *mūla* meaning ‘root’. Since this was done manually, a specific procedure is needed to work with it. This is as follows:

1. Mark the digits in the number as odd and even from right to left.
2. Pick the greatest possible square number which can be reduced from the last odd place (the leftmost one). For example, if the number is 8, the largest possible square which can be subtracted is the square of 2, which is 4.
3. Write the number (4 in the above example) in line with the odd digit and the square root of that number (2) separately. The first digit of the root is the digit which was squared.
4. Reduce the square.
5. Bring down the next digit.
6. Divide this number by twice the result written so far.
7. The quotient is the next digit of the root and is written to the right of the existing result. The remainder must be written below.
8. The next digit in the number will be brought down, to the right of the remainder.
9. Subtract from this number the square of the quotient (in step 7). The quotient must be chosen such that this subtraction is possible. If it is not possible, reduce the quotient by 1 and try again.
10. Bring down the next digit and divide by twice the result (step 6) again. Continue this process as long as there are still digits to the right.

The following is an example with 14642. In the first line, *o* stands for odd and *e* for even places.

	o	*e*	*o*	*e*	*o*	
	1	4	6	4	1	Result: 1
Reduce largest square	1					
	0					
Bring down next digit	0	4				
Divide by twice the result		2				
Remainder		0				Result: 1 2
Bring down next digit		0	6			

Reduce square of quotient	4			
	2			
Bring down next digit	2	4		
Divide by twice the result	2	4		
Remainder		0		Result: 1 2 1
Bring down the next digit		0	1	
Reduce square of quotient			1	
			0	

When dividing according to step 6, the quotients were small enough that they could be subtracted from the numbers in the later steps. Hence going back and repeating was not necessary.

If there is a remainder at the end of the process, and there are no digits left, the original number was not a square number. For example, 101 is not a square number but 100 is. If we applied the above process to 100, we will get 10 as the result and 0 as the remainder. But if we did the same with 101, the result will be 10 — the largest root that goes fully — and the remainder will be 1.

Cube

Ghana refers to cubing. One of the principal uses of *ghana* is in the computation of volume of solids or of spaces, a specific use case, unlike *varga* (square) which is more commonplace by comparison.

Ghana is defined as multiplying three instances of the number by each other. The first method for computing comes from this definition itself:

$$\begin{aligned} 5 \times 5 \times 5 &= 5 \times 25 \\ &= 125 \end{aligned}$$

Take three instances of the number and multiply them with each other to get the *ghana*.

The second method is to split the number into portions and compute the cube from there. The procedure for a two-digit number is as follows:

1. Take the first digit.
2. Write its cube.
3. Find 3 times the square of the first digit multiplied by the second digit.

4. Write this under the cube, shifted one place to the right.
5. Find 3 times the first digit multiplied by the square of the second digit.
6. Write this under the previous number, shifted one more place to the right.
7. Write the cube of the second digit, under the previous number, but shifted one more place to the right.
8. Add the numbers that you have written down to get the cube of the original number.

Let's see this with an example:

Take the number 15. Separately compute the following:
- 1^3
- $3 \times 1^2 \times 5$
- $3 \times 1 \times 5^2$
- 5^3

Then place them one below the other shifting each one by one place to the right as you place them. Finally, add the numbers. This procedure is shown below.

1^3	1			
$3 \times 1^2 \times 5$	1	5		
$3 \times 1 \times 5^2$		7	5	
5^3		1	2	5
	3	3	7	5

You might have recognised that this is an application of the well-known formula:

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

But the method as stated above can only be used with two-digit numbers. For using it with other numbers, we will need to do it repeatedly. Here's an example with 123.

1^3	1						
$3 \times 1^2 \times 23$		6	9				
$3 \times 1 \times 23^2$		1	5	8	7		
2^3				8			
$3 \times 2^2 \times 3$				3	6		
$3 \times 2 \times 3^2$					5	4	
3^3						2	7
	1	8	6	0	8	6	7

Notice what's different:

1. We're multiplying in the first iteration with the whole second portion of the number, i.e. 23.
2. We're shifting by 2 digits in each row in the first iteration, but by only 1 digit in the second iteration. This is because the second part is two digits long in the first iteration (23), but is only one digit long in the second iteration (3).
3. The second iteration overlaps with the first. More properly, the last row of the second iteration is two digits shifted from the previous row in the first iteration. Based on this, we need to calculate where the first row of the second iteration will come (i.e., 4 digits to the left of this point).

These techniques enable us to compute the cube easily by hand, as was the practice in earlier times.

One of the reasons for including *ghana* in the list of *parikarmas* is that it is used in a wide variety of practical situations. This is because finding volume was much more prominent in commerce and other day-to-day activities, as compared to today. Some practical uses from ancient times include:

- Working out the volume of grain (grain was usually measured in volume and not weight) and liquids
- Finding the size of water bodies like tanks and wells
- Finding the size of blocks of wood or stone used for construction

Cube Root

Ghanamūla refers to cube root. The procedure given for this is stated below. This is shown using 1728 as an example.

1. Mark each digit as a **cube** digit (units, thousands and every three digits from there), **first non-cube** (the one to the left of a cube) or **second non-cube** (the one to the right of a cube). [Note: the cube digit is also called *ghana* place]
2. From the last *ghana* place (the leftmost one), subtract the largest possible cube number (1, 8, 27, …, 729).
3. Take the cube root of this cube number. (This will be between 1 and 9). Place it on a row to the right. This row is for writing the result.
4. Bring the next second non-cube digit.
5. Multiply the square of the result (from the row) so far by three and divide the value (7 in the example) by this.
6. Note the remainder under the number and the quotient to the right.
7. Bring down the next digit (which is the first non-cube).
8. Multiply the result accumulated thus far by the square of the quotient kept on the right. Multiply this by three and subtract from the value. (See step 11 if subtraction isn't possible.)
9. Bring down the next digit (which is the next cube digit).
10. Reduce the cube of the quotient.
11. If subtraction in steps 8 and 10 was feasible (either zero or a positive number), mark the quotient as the next digit of the result. It will now be part of the result. If either of those steps failed (negative answer), reduce the quotient in step 5 by 1 and repeat the steps that follow. The remainder will increase accordingly.
12. Repeat steps 4 to 11 until all digits of the number are complete.

	c	s	f	c		
	1	7	2	8	Result:	1
Reduce largest cube	1					
	0					
Bring down the next digit	0	7				
Divide by thrice of result squared		3			Quotient:	2
Remainder		1				
Bring down the next digit		1	2			
Reduce thrice quotient squared times result		1	2			
			0			
Bring down the next digit			0	8		
Reduce quotient cubed				8		
Subtraction worked. Add quotient to result.				0	Result:	12

The primary use of cube root is to find the side length that created a specific volume, for example the side length of a cubic vessel that holds a specific volume of water.

Questions

1. Multiply 5869 by 10298 using the *kapāṭasandhi* method.
2. Find the square of:
   - 6348
   - 9989
3. Find the square root of:
   - 16384
   - 906305
4. Find the cube of 345.
5. Find the cube root of 941192.

Chapter 3

Fractions & Zero

Purpose of Fractions

In Indian mathematics, fractions or *bhinna* were the chief way in which parts were represented. Decimals were not used, and are not mentioned in major texts.

Given this importance of fractions, knowledge of how to compute with them was of crucial importance. Consequently, this was one of the first topics discussed in many books.

Computing with fractions is straightforward — we learn much of it in school already. What you'll find below is a list of terms used in Sanskrit when dealing with fractions, along with their English translations:

1. *bhinna* - fraction
2. *bhāga* - fraction or division or portion
3. *aṃśa* - numerator
4. *hāra* - denominator
5. *cheda* - denominator
6. *rūpa* - integer (either an integer without fractional parts, or the integer part of a mixed number)
7. *samaccheda* - having equal denominator (a necessary condition for addition or subtraction between fractions)
8. *apavartana* - reducing (simplifying) fractions (this can be done partially or be brought to simplest terms, and was generally optional)
9. *khahara* - a special fraction where 0 is the denominator; this relates to the concept of division by zero which is treated today as undefined or as infinity in some contexts.

Notation

Indian mathematicians wrote fractions similar to how we write them today but without the bar between the numerator and denominator.

$\frac{1}{2}$ was written as $\begin{matrix} 1 \\ 2 \end{matrix}$

Mixed numbers were written such that the *rūpa* or integer part was at the top, followed by the *aṃśa* (numerator) and then the *cheda* (denominator), one below the other.

$2\frac{1}{3}$ would be written $\begin{matrix} 2 \\ 1 \\ 3 \end{matrix}$

Operations with Fractions

Like *rūpas* that we encountered in Chapter 2, *bhinnas* are also subject to *parikarmas.*

- Addition and subtraction: the fractions have to be made *samaccheda.* The new numerators are the old numerators multiplied by the opposite denominator. The new denominator is the product of the two old denominators.

$$\begin{aligned} \frac{1}{2} + \frac{1}{3} &= \frac{1 \times 3}{2 \times 3} + \frac{1 \times 2}{2 \times 3} \\ &= \frac{3}{6} + \frac{2}{6} \qquad \text{(samaccheda)} \\ &= \frac{5}{6} \end{aligned}$$

- Multiplication and division: directly multiply the numerator with the numerator and denominator with denominator. Example:

$$\frac{1}{2} \times \frac{5}{6} = \frac{1 \times 5}{2 \times 6} = \frac{5}{12}$$

- The other four *parikarmas*: Apply separately on the numerator and denominator. Example:

$$\left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2} = \frac{4}{9}$$

Zero

Zero helps us represent the absence of a quantity effectively. Zero is also subject to *parikarmas.* These are all well known from school, hence they are not dealt with here.

When something is divided by zero, it becomes a *khahara*, a special fraction with 0 as its denominator. These are usually considered equal to infinity for practical purposes.

Negative numbers

Negative numbers are called *ṛṇa*, which literally means ‘debt’. Positive numbers are called *dhana*, which means ‘wealth’.

Negative numbers are usually represented with a dot over the number:

$\dot{3}$ means -3

This can also be done with fractions:

$\begin{matrix}\dot{2}\\3\end{matrix}$ means $-\frac{2}{3}$

The same applies to mixed numbers as well:

$\begin{matrix}\dot{1}\\2\\5\end{matrix}$ means $-1\frac{2}{5}$

But note what happens when the dot goes over the middle number of the mixed number:

$\begin{matrix}2\\\dot{1}\\2\end{matrix}$ means $2-\frac{1}{2}=1\frac{1}{2}$

Some writers also used a ‘+’ symbol for clarity as a dot might be too small to notice.

Questions

1. Write the following in Western notation:
   - $\begin{matrix}4\\5\end{matrix}$
   - $\begin{matrix}2\\4\\5\end{matrix}$
   - $\dot{2}$
   - $\begin{matrix}\dot{2}\\5\\9\end{matrix}$
   - $\begin{matrix}3\\\dot{1}\\2\end{matrix}$
2. Write the following in Indian notation:
   - $\frac{2}{3}$
   - $\frac{9}{2}$
   - $2\frac{1}{3}$

- -98
- $1\frac{5}{6}$ (*Hint: There are two ways to write this.*)

Chapter 4

Inverse Operations

Inverse operations refers to doing the operations in reverse order. This is called *vyastavidhi* in Sanskrit.

Let's say we have a situation where we're doing multiple *parikarmas* in order:

Number ⟶ Multiply by 2 ⟶ Add 3 ⟶ Result

In our specific situation, we know the result already and we want to find the original number. This is where *vyastavidhi* is useful.

The rule of *vyastavidhi* says that we replace each operation by its opposite operation and reverse the order in which the *parikarmas* were applied.

The opposites are:

Operation	Opposite
Addition	Subtraction
Subtraction	Addition
Multiplication	Division
Division	Multiplication
Square	Square root
Square root	Square
Cube	Cube root
Cube root	Cube

Let's apply this to the example above. The opposite of multiplication is division. We have to divide by the same number, that is, 2. The opposite of addition of 3, is subtraction of 3. In the reverse order, we get:

Number ⟵ Divide by 2 ⟵ Subtract 3 ⟵ Result

A real-world example

A boat carrying passengers went through a river, stopping at various places. The boatman departed from his hometown with some passengers. You heard the boatman mention that: at the first stop, 3 people got in; at the second stop, half the boat emptied; at the third stop, 5 people got down. You were standing at the fourth stop and observed that the boat arrived with 5 people. How many people did it have when it first departed?

Operations in order:

- Add 3
- Divide by 2
- Subtract 5

Result is 5.

Apply *vyastavidhi*: opposite operations in reverse order:

- Add 5
- Multiply by 2
- Subtract 3.

This is done to the result which is 5. Answer is 17.

The boat originally had 17 people when the boatman departed.

Question

The Orange Saga: I had a few oranges in my home last Saturday. On Wednesday, I made a deal with my mathematician friend to keep the square root of whatever I had that day and give him the rest. My mother bought me 2 last Sunday. I visited the market on Friday and purchased 3. I gave 5 to my friend on Monday. Today is Saturday and I have 5 oranges. How many did I have one week ago?

Chapter 5

Rule of Three

Trairāśika is the Indian version of ratio and proportion. This is often translated as "rule of three", though it literally means "operation with three numbers".

You'll find it in nearly every book of mathematics that introduces mathematics from the basic concepts. The importance of *trairāśika* in Indian maths cannot be overstated. It occupies a foundational role in nearly everything that follows and is arguably the most important tool that you should possess if you wanted to study the subject further.

Trairāśika is called so because, in it, we deal with three numbers or *rāśis*. When we want to point at a specific number, such as when a number is being used in calculations, it is usually called *rāśi* in Sanskrit. This is opposed to the more abstract concept of numbers or number systems, where number would be called *saṅkhyā*. The three numbers of *trairāśika* are:

1. *pramāṇa* (also called *māna*)
2. *pramāṇaphala* (also called *phala*)
3. *icchā*

An example would illustrate these clearly:

3 mangoes cost 6 *paṇas*, how much do 5 mangoes cost?

3 mangoes are the *pramāṇa*. The *phala* is 6 *paṇas* (*paṇa* was a currency in pre-colonial India). 5 mangoes are the *icchā*. Notice that the entire problem only has 3 numbers. This is why it is called *trairāśika*.

The answer to the above question is what we want to know, which is termed *icchāphala*. We get that by multiplying the *phala* with the *icchā* and then dividing by the *pramāṇa*.

In the above example, it is 6 times 5 divided by 3, which is 10 *paṇas*.

Trairāśika is used extensively in several situations such as:

- Fractional computation (since Indians did not use decimals)
- Geometry: cases like similar triangles or enlarging or shrinking shapes

- Commerce: barter, sale and purchase, fixing prices for commodities, and so on

Questions

1. The price of lemons is 10 for 2 *paṇas*. How much money should I carry if I want 7?

2. At the market, there are two chair vendors selling the same type of chair. One said he will sell 13 for 6 *paṇas*. The other said he will sell 6 for 2 *paṇas*. Which vendor sells it cheaper?

3. One merchant has 90 sheep and can sell 20 of them at the rate of 3 for 2 *paṇas*. He needs cows and saw another merchant selling cows at the rate of 4 for 3 *paṇas*. He has appointed you as his mathematician to tell him how many sheep he should sell if he wants ten cows. Can you advise him?

Chapter 6

Sequences

Śreḍhī literally means 'ladder' in Sanskrit. In mathematics, it refers to sequences. Any sequence in mathematics, informally speaking, is a set of numbers in a specific order where there is a fixed pattern for how you get from one number to the next. Indian texts generally dealt with two major kinds of sequences:

1. *Yathottaraśreḍhī* - arithmetic sequences
2. *Guṇottaraśreḍhī* - geometric sequences

In arithmetic sequences, you add or subtract a number. In geometric sequences, you multiply or divide by another number.

Arithmetic Sequences

Arithmetic sequences are when the starting number is fixed and every number that follows, adds or subtracts a fixed number from the previous one. This is clear from the name: *yathottaraśreḍhī* is split as *yathā-uttara-śreḍhī* — which means a ladder (*śreḍhī*) that moves upwards (*uttara*) as defined earlier (*yathā* = as stated).

The starting number is called *ādi* or *mukha*. The fixed number that is added is called *caya*. The number of terms in the entire sequence is called *gaccha* or *pada*.

Some examples include:

śreḍhī	*ādi*	*caya*
1, 3, 5, 7, …	1	2
10, 6, 2	10	$\dot{4}$
3, 8, 13, 18	3	5

As seen above, there is a fixed *mukha* and a fixed *caya* for each sequence. Some sequences, such as the first one shown above, might be unending, while others, such as the next two examples given above, end after a few instances.

In the context of sequences, the word *dhana* is used to denote the value of a specific term. Furthermore:

- *madhyadhana* (middle-value) refers to the sequence's middle term
- *antyadhana* (last-value) refers to the last term in the sequence
- *sarvadhana* (total-value) refers to the sum of all terms of the sequence. This is also called the *śreḍhīphala* which means 'result of the sequence'.

Procedure

Here's how these are calculated:

The *gaccha* reduced by one, is multiplied by the *caya*, and the *mukha* is added to this to get *antyadhana*. Add the *mukha* to this and halve it to get *madhyadhana*. Multiply the latter by *gaccha* to get *sarvadhana*, which is also called *gaṇita*.

The *parikarmas* done here are:

$$\boxed{\textit{gaccha}} \longrightarrow \begin{array}{c}\text{Subtract}\\ 1\end{array} \longrightarrow \begin{array}{c}\text{Multiply by}\\ \textit{caya}\end{array} \longrightarrow \begin{array}{c}\text{Add}\\ \textit{mukha}\end{array} \longrightarrow \boxed{\textit{antyadhana}}$$

$$\boxed{\textit{antyadhana}} \longrightarrow \begin{array}{c}\text{Add}\\ \textit{mukha}\end{array} \longrightarrow \begin{array}{c}\text{Divide by}\\ 2\end{array} \longrightarrow \boxed{\textit{madhyadhana}}$$

$$\boxed{\textit{madhyadhana}} \longrightarrow \begin{array}{c}\text{Multiply by}\\ \textit{gaccha}\end{array} \longrightarrow \boxed{\textit{sarvadhana}}$$

For ease of reference, Western-style equations are also given:

$$(\text{gaccha} - 1) \times \text{caya} + \text{mukha} = \text{antyadhana}$$
$$\frac{\text{antyadhana} + \text{mukha}}{2} = \text{madhyadhana}$$
$$\text{madhyadhana} \times \text{gaccha} = \text{sarvadhana}$$

As we can see, the calculations are quite easy, and it only needs some practice to master this.

Texts often mention multiple ways to arrive at each of these, for example, you're given *sarvadhana*, *gaccha* and *mukha*, how do you compute *caya* with that? A number of similar combinations are discussed and the formulae for these are also given. These can be derived through *vyastavidhi* from the above.

Geometric Sequences

Geometric sequences occur when we multiply or divide by a fixed number. This is also clear from the name: *guṇa* means 'multiplication'.

Like arithmetic sequences, geometric sequences also have an *ādi* or *mukha*, which is the first number in the sequence. There is also a *gaccha* or *pada* which is the number of terms. In place of *caya*, we have *uttara*, which is the number with which we multiply to get the next term.

Some examples of geometric sequences:

śreḍhī	*ādi*	*uttara*
1, 2, 4, 8, 16	1	2
1000, 200, 40, 8	1000	$\frac{1}{5}$
27, 36, 48, 64	27	$\frac{4}{3}$

When we multiply the previous number by a fixed value each time, the common multiplier is a whole number. If we divide it each time, the common multiplier is a reciprocal, such as in the second example above where the multiplier is $\frac{1}{5}$ when each term is the previous term divided by 5. We can also have more complex multipliers such as shown in the third example.

Sequence's sum

One of the main concerns in sequences is to find the sum of all terms. We saw the procedure for *sarvadhana* in arithmetic sequences earlier.

For geometric sequences, the steps for the sum are as follows:
1. Take the *gaccha*
2. If it is odd, subtract 1 — mark it as **m**. If it is even, divide by 2 — mark it as **s**.
3. Continue doing the above step to the result until the value becomes 0, writing at each step, the letters *m* and *s* in order, depending on whether subtraction or division was done.
4. Start with the number 1 and read the sequence of letters that you wrote backwards. Whenever you encounter *s*, square the number at hand. Whenever it is *m*, multiply it by the *uttara*. Do this until you cover the entire set of letters.
5. Reduce one from the above result.
6. Mutliply by *mukha*.

Let's take the first example (1, 2, 4, 8, 16) and find the sum:

1. *gaccha* is 5.

2. 5 is odd. $5 - 1 = 4$. We note 'm'.

3. Procedure:

5	m
4	s
2	s
1	m
0	-

   Sequence: *mssm*

4. Apply in reverse:

   m: multiply by *uttara* which is 2
   s: square

   $$1 \xrightarrow{\text{m}} 2 \xrightarrow{\text{s}} 4 \xrightarrow{\text{s}} 16 \xrightarrow{\text{m}} 32$$

5. Reduce 1: 31

6. Multiply by *mukha*, which is 1: 31

Let's check the result: $1 + 2 + 4 + 8 + 16 = 31$ — result is correct.

Uses

Computation with *śreḍhī* appears prominently in commercial problems, such as those pertaining to interest or payments that increase over time.

Questions

1. Find the *antyadhana* of an arithmetic sequence where *mukha*, *caya* and *gaccha* are 3, 2 and 7 respectively.

2. Find the *sarvadhana* of the first ten odd numbers.

3. Find the *sarvadhana* of 2, 6, 18, ..., 1458.

4. A doctor employed in a hospital found that the number of patients yesterday was 200. The number of patients today is 192. How many patients should he expect 10 days hence?

5. A merchant lent money at 1% interest per month. He lent 100 *paṇas* to one client. How much should he get in one year?

Glossary

Provided below is a list of Sanskrit terminology used in the chapters so far along with the meaning of the words. The list is provided in Sanskrit order (a, ā, i, ...).

aṃśa numerator, part, portion
antara difference
antya the 16th position from the right in the place value system (10^{15})
antyadhana last term of a sequence
apavartana reducing (simplifying) fractions (this can be done partially or be brought to simplest terms, and is optional)
abda the 10th position from the right in the place value system (10^{9})
abhyāsa product
ayuta ten thousand, the 5th position from the right in the place value system (10^{4})
arbuda the 9th position from the right in the place value system (10^{8})
ādi first term of an arithmetic or geometric sequence
āhati product
icchā third term in a *trairāśika.*
icchāphala the answer of a *trairāśika.*
uttara the fixed number by which one term of a geometric sequence is multiplied to get the next term (common multiplier)
ūna subtracted
ṛṇa debt (literal meaning), negative number
eka one, the 1st position from the right in the place value system (10^{0})
kapāṭasandhi the door-joint method (a method of multiplying two numbers)
koṭi crore, the 8th position from the right in the place value system (10^{7})
kharva the 11th position from the right in the place value system (10^{10})
khahara a special fraction where 0 is the denominator (treated as infinity in most practical contexts)
gaccha number of terms in a sequence
guṇa multiplication
guṇakāra multiplication
guṇaka the number with which it is multiplied (multiplier)
guṇottaraśreḍhī geometric sequence
guṇya the number to be multiplied (multiplicand)

ghana cube (operation)
ghanamūla cube root
caya the fixed number added in an arithmetic sequence to one term to get the next term (common difference)
cheda denominator
jaladhi the 15th position from the right in the place value system (10^{14})
nikharva the 12th position from the right in the place value system (10^{11})
taṣṭa divide and take the remainder (modulo)
trairāśika rule of three, ratio
daśa ten, the 2nd position from the right in the place value system (10^{1})
dhana wealth (literal meaning), positive number, value
madhya middle, the 17th position from the right in the place value system (10^{16})
madhyadhana middle term of a sequence
mahāpadma the 13th position from the right in the place value system (10^{12})
mukha first term of an arithmetic or geometric sequence
paṇa a unit of currency from pre-colonial India
pada number of terms in a sequence
parārdha the 18th position from the right in the place value system (10^{18})
parikarma basic operation, one of the eight operations recognised as foundational in Indian mathematics
purāṇas a class of Sanskrit literature that is encyclopedic in nature, covering a range of different subjects.
pramāṇa first term in a *trairāśika* (also abbreviated as *māna*).
pramāṇaphala second term in a *trairāśika* (also abbreviated as *phala*).
prayuta million, the 7th position from the right in the place value system (10^{6})
phala result of an operation, quotient, second term in a *trairāśika*, sum of a sequence
bhakta divided
bhāga fraction, division, portion
bhinna fraction, mixed number (a number which has a whole and fractional part, like $1\frac{1}{2}$)
yathottaraśreḍhī arithmetic sequence
yuti addition (the act of adding two numbers)

yoga sum

rāśi number (definitive)

rūpa integer (either an integer without fractional parts, or the integer part of a mixed number)

lakṣa lakh, the 6th position from the right in the place value system (10^5)

varga square (operation)

vargamūla square root

viyuta subtracted

viyuti subtraction (the act of subtracting)

vyavakalita subtraction (the operation)

vyastavidhi inverse operations

śaṅkha the 14th position from the right in the place value system (10^{13})

śata the 3rd position from the right in the place value system (10^2)

śeṣa remainder

śreḍhī ladder (literal meaning), sequence

śreḍhīphala sum of all terms of a sequence

saṃvarga product

saṅkalita addition (as an operation)

saṅkhyā number (abstract or definitive)

samaccheda having equal denominator (a necessary condition for addition or subtraction between fractions)

sarvadhana sum of all terms of a sequence

sahasra the 4th position from the right in the place value system (10^3)

hata multiplied

haraṇa division

hāra denominator

hṛta divided